14-1 Graphing Trigonometric Functions

Main Ideas

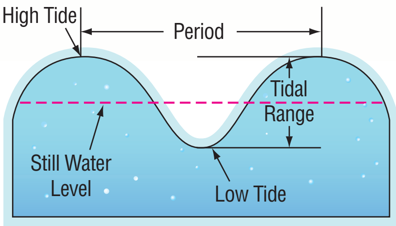
- Graph trigonometric functions.
- Find the amplitude and period of variation of the sine, cosine, and tangent functions.

New Vocabulary

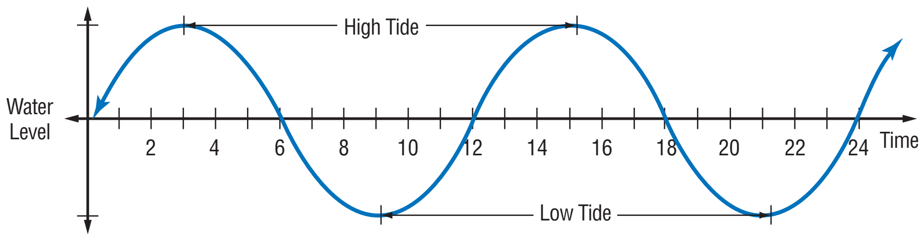
amplitude

GET READY for the Lesson

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.



Graph Trigonometric Functions The diagram below illustrates the water level as a function of time for a body of water with semidiurnal tides.



Review Vocabulary

Period The least possible value of a for which $f(x) = f(x + a)$.

In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is said to be *periodic*.

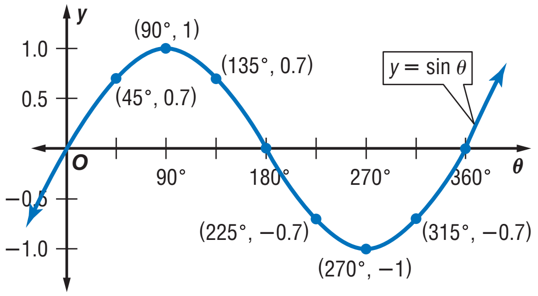
To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

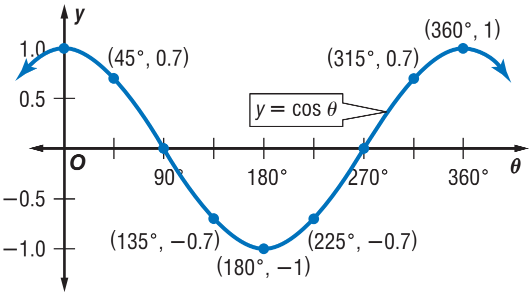
To graph the functions $y = \sin \theta$, $y = \cos \theta$, or $y = \tan \theta$, use values of θ expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form $(\theta, \sin \theta)$, $(\theta, \cos \theta)$, and $(\theta, \tan \theta)$, respectively.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
nearest tenth	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
nearest tenth	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0	0.5	0.7	0.9	1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0
nearest tenth	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

nd = not defined

After plotting several points, complete the graphs of $y = \sin \theta$ and $y = \cos \theta$ by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or 2π radians. That is, the graph of each function repeats itself every 360° or 2π radians.





Notice that both the sine and cosine have a maximum value of 1 and a minimum value of −1. The **amplitude** of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is $\left|\frac{1-(-1)}{2}\right|$ or 1.

Concepts in Motion
Animation
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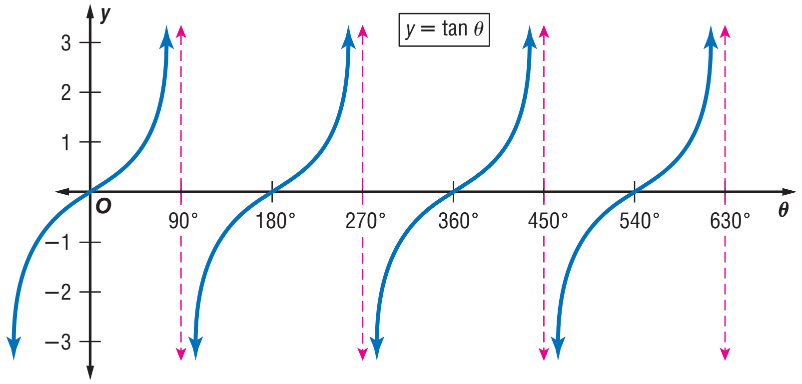
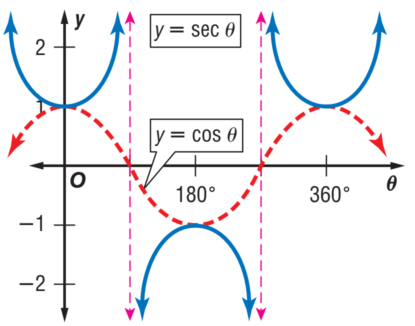
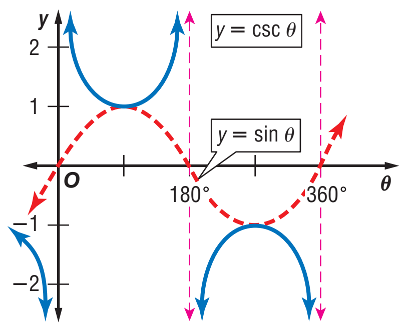
By examining the values for tan θ in the table, you can see that the tangent function is not defined for 90°, 270°, …, 90° + $k \cdot$ 180°, where k is an integer. The graph is separated by vertical asymptotes whose x-intercepts are the values for which $y = \tan \theta$ is not defined.

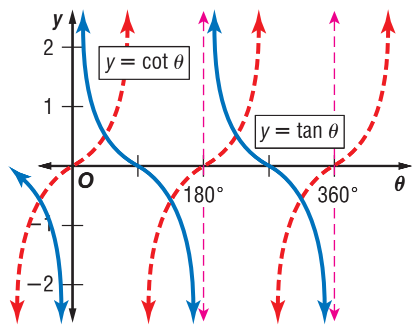


The period of the tangent function is 180° or π radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

Compare the graphs of the secant, cosecant, and cotangent functions to the graphs of the cosine, sine, and tangent functions, shown below.







Notice that the period of the secant and cosecant functions is 360° or 2π radians. The period of the cotangent is 180° or π radians. Since none of these functions have a maximum or minimum value, they have no amplitude.

Variations of Trigonometric Functions Just as with other functions, a trigonometric function can be used to form a family of graphs by changing the period and amplitude.

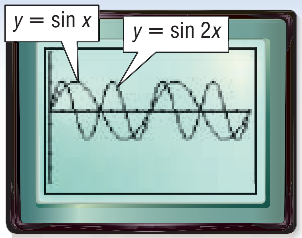
GRAPHING CALCULATOR LAB

Period and Amplitude

On a TI-83/84 Plus, set the MODE to degrees.

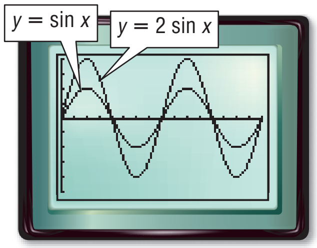
THINK AND DISCUSS

1. Graph $y = \sin x$ and $y = \sin 2x$. What is the maximum value of each function?
2. How many times does each function reach a maximum value?
3. Graph $y = \sin\left(\frac{x}{2}\right)$. What is the maximum value of this function? How many times does this function reach its maximum value?



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

4. Use the equations $y = \sin bx$ and $y = \cos bx$. Repeat Exercises 1–3 for maximum values and the other values of b. What conjecture can you make about the effect of b on the maximum values and the periods of these functions?
5. Graph $y = \sin x$ and $y = 2 \sin x$. What is the maximum value of each function? What is the period of each function?
6. Graph $y = \frac{1}{2} \sin x$. What is the maximum value of this function? What is the period of this function?



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7. Use the equations $y = a \sin x$ and $y = a \cos x$. Repeat Exercises 5 and 6 for other values of a. What conjecture can you make about the effect of a on the amplitudes and periods of $y = a \sin x$ and $y = a \cos x$?

Study Tip

Amplitude and Period

Note that the amplitude affects the graph along the vertical axis and the period affects it along the horizontal axis.

The results of the investigation suggest the following generalization.

KEY CONCEPT — ***Amplitudes and Periods***

Words For functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
For functions of the form $y = a \tan b$, the amplitude is not defined, and the period is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

Examples

$y = 3 \sin 4\theta$	amplitude 3 and period $\frac{360°}{4}$ or $90°$
$y = -6 \cos 5\theta$	amplitude $\lvert -6 \rvert$ or 6 and period $\frac{2\pi}{5}$
$y = 2 \tan \frac{1}{3}\theta$	no amplitude and period 3π

You can use the amplitude and period of a trigonometric function to help you graph the function.

EXAMPLE 1 Graph Trigonometric Functions

Find the amplitude, if it exists, and period of each function. Then graph the function.

a. $y = \cos 3\theta$

First, find the amplitude.

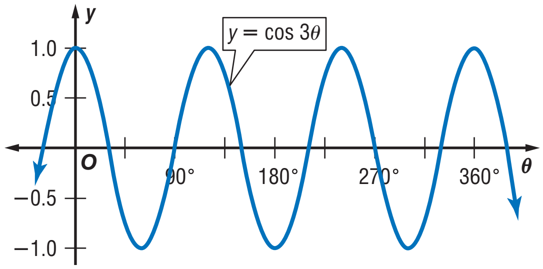
$|a| = |1|$ The coefficient of $\cos 3\theta$ is 1.

Next, find the period.

$\frac{360°}{|b|} = \frac{360°}{|3|}$ $b = 3$

$= 120°$

Use the amplitude and period to graph the function.

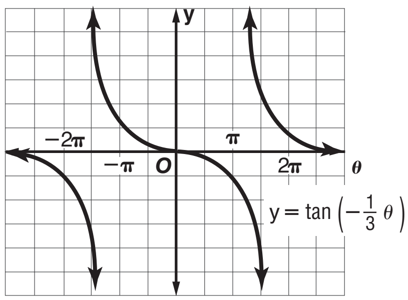


b. $y = \tan\left(-\frac{1}{3}\theta\right)$

Amplitude: This function does not have an amplitude because it has no maximum or minimum value.

Period: $\frac{\pi}{|b|} = \frac{\pi}{\left|-\frac{1}{3}\right|}$

$= 3\pi$



Study Tip

Amplitude

Notice that the graph of the longest function has no amplitude, because the tangent function has no minimum or maximum value.

CHECK Your Progress

1A. $y = \frac{1}{4} \sin \theta$

1B. $y = -2 \sec\left(\frac{1}{4}\theta\right)$

Real-World EXAMPLE Use Trigonometric Functions

2 OCEANOGRAPHY Refer to the application at the beginning of the lesson. Suppose the tidal range of a city on the Atlantic coast is 18 feet. A tide is at *equilibrium* when it is at its normal level, halfway between its highest and lowest points. Write a function to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$ and that the high tide is beginning. Then graph the function.

Since the height of the tide is 0 at $t = 0$, use the sine function $h = a \sin bt$, where a is the amplitude of the tide and t is time in hours.

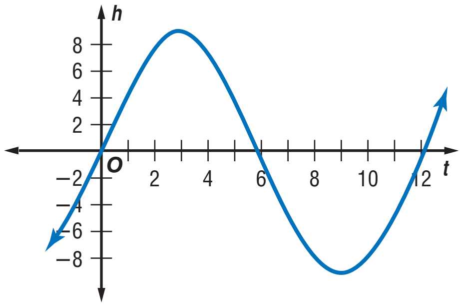
Find the amplitude. The difference between high tide and low tide is the tidal range or 18 feet.

$a = \frac{18}{2}$ or 9

Find the value of b. Each tide cycle lasts about 12 hours.

$\frac{2\pi}{|b|} = 12$ period $= \frac{2\pi}{|b|}$

$b = \frac{2\pi}{12}$ or $\frac{\pi}{6}$ Solve for b.



Thus, an equation to represent the height of the tide is $h = 9 \sin \frac{\pi}{6}t$.

Real-World Link

Lake Superior has one of the smallest tidal ranges. It can be measured in inches, while the tidal range in the Bay of Fundy in Canada measures up to 50 feet.

Source: Office of Naval Research

CHECK Your Progress

2A. Assume that the tidal range is 13 feet. Write a function to represent the height h of the tide. Assume the tide is at equilibrium at $t = 0$ and that the high tide is beginning.

2B. Graph the tide function.

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CHECK Your Understanding

Example 1 (p. 825)

Find the amplitude, if it exists, and period of each function. Then graph each function.

1. $y = \frac{1}{2} \sin \theta$ **2.** $y = 2 \sin \theta$ **3.** $y = \frac{2}{3} \cos \theta$

4. $y = \frac{1}{4} \tan \theta$ **5.** $y = \csc 2\theta$ **6.** $y = 4 \sin 2\theta$

Example 2 (p. 826)

7. $y = 4 \cos \frac{3}{4}\theta$ **8.** $y = \frac{1}{2} \sec 3\theta$ **9.** $y = \frac{3}{4} \cos \frac{1}{2}\theta$

BIOLOGY For Exercises 10 and 11, use the following information.
In a certain wildlife refuge, the population of field mice can be modeled by $y = 3000 + 1250 \sin \frac{\pi}{6}t$, where y represents the number of mice and t represents the number of months past March 1 of a given year.

10. Determine the period of the function. What does this period represent?

11. What is the maximum number of mice, and when does this occur?

Exercises

HOMEWORK HELP

For Exercises	See Examples
12–23	1
24–26	2

Find the amplitude, if it exists, and period of each function. Then graph each function.

12. $y = 3 \sin \theta$

13. $y = 5 \cos \theta$

14. $y = 2 \csc \theta$

15. $y = 2 \tan \theta$

16. $y = \frac{1}{5} \sin \theta$

17. $y = \frac{1}{3} \sec \theta$

18. $y = \sin 4\theta$

19. $y = \sin 2\theta$

20. $y = \sec 3\theta$

21. $y = \cot 5\theta$

22. $y = 4 \tan \frac{1}{3}\theta$

23. $y = 2 \cot \frac{1}{2}\theta$

MEDICINE **For Exercises 24 and 25, use the following information.**

Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

24. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

25. How do the periods of the tuning forks compare?

Find the amplitude, if it exists, and period of each function. Then graph each function.

26. $y = 6 \sin \frac{2}{3}\theta$

27. $y = 3 \cos \frac{1}{2}\theta$

28. $y = 3 \csc \frac{1}{2}\theta$

29. $y = \frac{1}{2} \cot 2\theta$

30. $2y = \tan \theta$

31. $\frac{3}{4}y = \frac{2}{3} \sin \frac{3}{5}\theta$

32. Draw a graph of a sine function with an amplitude $\frac{3}{5}$ and a period of 90°. Then write an equation for the function.

33. Draw a graph of a cosine function with an amplitude of $\frac{7}{8}$ and a period of $\frac{2\pi}{5}$. Then write an equation for the function.

34. Graph the functions $f(x) = \sin x$ and $g(x) = \cos x$, where x is measured in radians, for x between 0 and 2π. Identify the points of intersection of the two graphs.

35. Identify all asymptotes to the graph of $g(x) = \sec x$.

BOATING **For Exercises 36–38, use the following information.**

A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

EXTRA PRACTICE
See pages 922, 939.
Math Online
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36. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

37. Draw a graph showing the height of the buoy as a function of time.

38. What is the height of the buoy after 12 seconds?

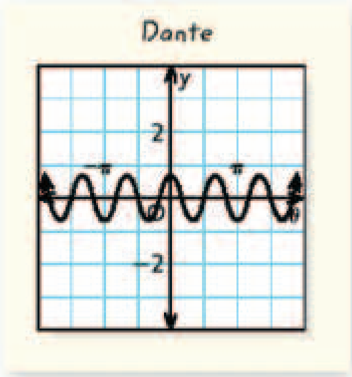
H.O.T. Problems

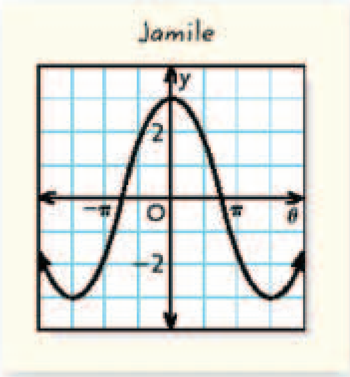
39. **OPEN ENDED** Write a trigonometric function that has an amplitude of 3 and a period of π. Graph the function.

40. **REASONING** Explain what it means to say that the period of a function is 180°.

41. **CHALLENGE** A function is called *even* if the graphs of $y = f(x)$ and $y = f(-x)$ are exactly the same. Which of the six trigonometric functions are even? Justify your answer with a graph of each function.

42. **FIND THE ERROR** Dante and Jamile graphed $y = 3 \cos \frac{2}{3}\theta$. Who is correct? Explain your reasoning.

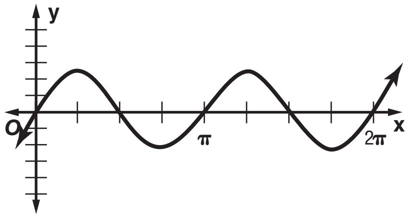




43. *Writing in Math* Use the information on page 822 to explain how you can predict the behavior of tides. Explain why certain tidal characteristics follow the patterns seen in the graph of the sine function.

STANDARDIZED TEST PRACTICE

44. **ACT/SAT** Identify the equation of the graphed function.



A $y = \frac{1}{2} \sin 4\theta$ **C** $y = 2 \sin \frac{1}{4}\theta$

B $y = \frac{1}{4} \sin 2\theta$ **D** $y = 4 \sin \frac{1}{2}\theta$

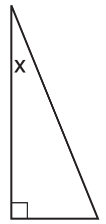
45. **REVIEW** Refer to the figure below. If $\tan x = \frac{10}{24}$, what are $\sin x$ and $\cos x$?

F $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$

G $\sin x = \frac{10}{26}$ and $\cos x = \frac{24}{26}$

H $\sin x = \frac{26}{10}$ and $\cos x = \frac{26}{24}$

J $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$



Spiral Review

Solve each equation. (Lesson 13-7)

46. $x = \text{Sin}^{-1} 1$ **47.** $\text{Arcsin}(-1) = y$ **48.** $\text{Arccos} \frac{\sqrt{2}}{2} = x$

Find the exact value of each function. (Lesson 13-6)

49. $\sin 390°$ **50.** $\sin(-315°)$ **51.** $\cos 405°$

52. **PROBABILITY** There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. (Lesson 12-5)

53. Find the first five terms of the sequence in which $a_1 = 3$, $a_{n+1} = 2a_n + 5$. (Lesson 11-5)

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each pair of functions on the same set of axes.** (Lesson 5-7)

54. $y = x^2, y = 3x^2$ **55.** $y = 3x^2, y = 3x^2 - 4$ **56.** $y = 2x^2, y = 2(x + 1)^2$